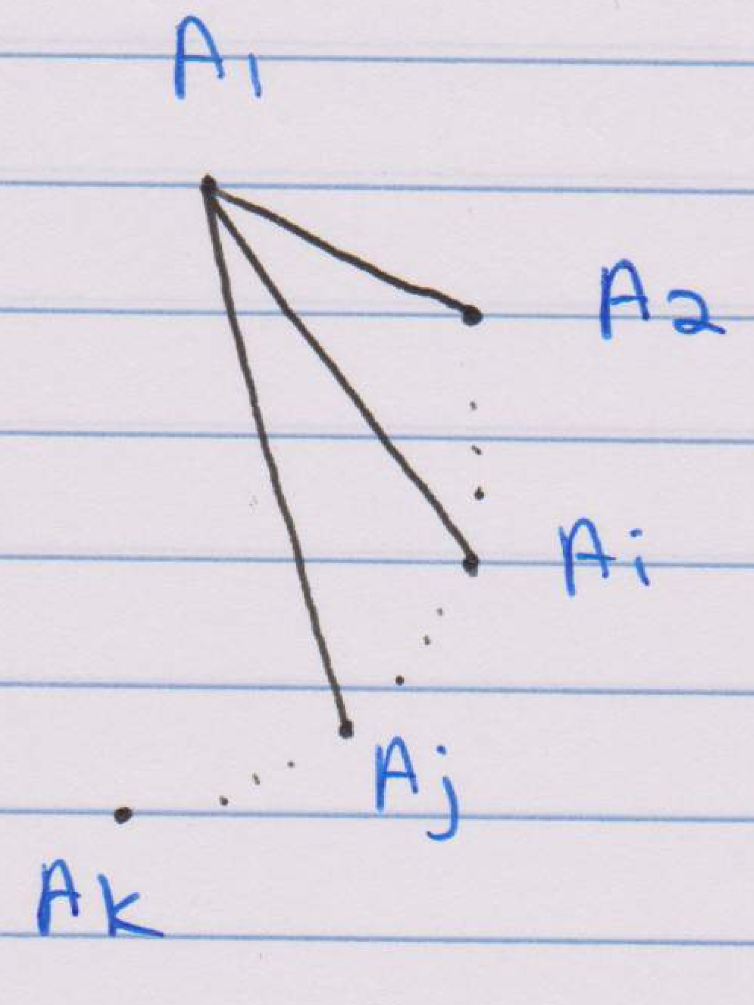
Principle of Extremal Example

Consider n points on a plane s.t. every point is connected via edges with at least 3 other points. Show that there must exist a cycle with an even number of edges.

Soln:

Consider the longest path which does not revisit a vertex twice in our finite graph: $A_1 \to A_2 \to \dots A_k$.

By assumption, the point A_1 must be connected to at least 3 other points, one of which is A_2. The other 2 points must be points of the path or otherwise we would be able to create an even longer path, contradicting the fact that we considered the longest path. Let's assume that A_1 is connected with A_i and A_j s.t. $i<j$.



We have the following 3 cycles:

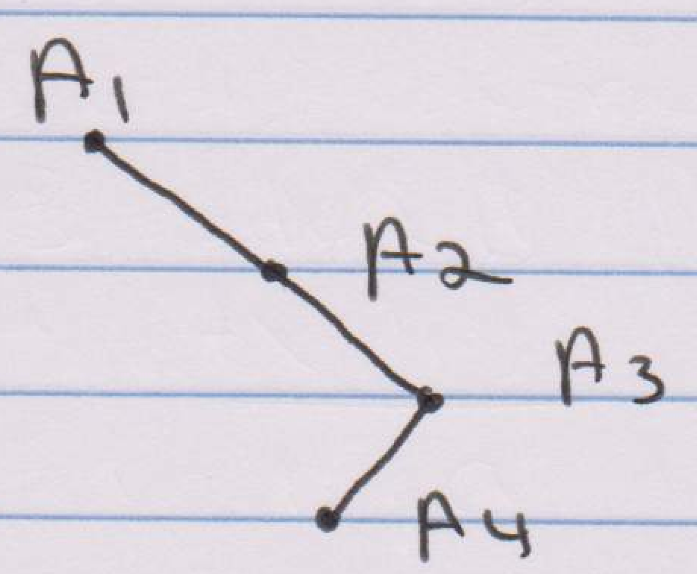
Cycle 1: $A_1 \rightarrow A_2 \rightarrow \ldots A_i \rightarrow \ldots A_j \rightarrow A_1$

Cycle 2: $A_1 \rightarrow A_2 \rightarrow \ldots A_i \rightarrow A_1$

Cycle 3: $A_1 \rightarrow A_i \rightarrow \ldots A_j \rightarrow A_1$

If i is even, then cycle 2 is even. This is because from $A_1 \rightarrow A_2 \rightarrow \ldots A_i$, there are $i-1$ edges. However, since there is another edge from A_i to A_1, there are i edges in total.

Remark: To show why there are $i-1$ edges from $A_1 \rightarrow A_2 \rightarrow \ldots A_i$, consider the following diagram below:



From $A_1 \rightarrow A_2$, there is 1 edge.
From $A_1 \rightarrow A_2 \rightarrow A_3$, there are 2 edges.
From $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4$, there are 3 edges.

If j is even, then cycle 1 is even based on the logic applied above.

If both i and j are odd, then cycle 3 is even since its length is $j-i+2$.

Remark: To explain why cycle 3 has length , consider the following below:

$j-i+2$

a) $A_1 \to$ A_i has length 1.

b) $A_i \to \ldots A_j$ has length $j-i$.

To prove this, consider the following:

$A_1 \to \ldots A_i = i-1$

$A_1 \to \ldots A_j = j-1$

$$\begin{aligned} A_i \to \ldots A_j &= (A_1 \to \ldots A_j) - (A_1 \to \ldots A_i) \\ &= (j-1) - (i-1) \\ &= j-i \end{aligned}$$

c) $A_j \to A_1$ has length 1.

Putting it all together, we get $j-i+2$.